Exact calculation of the spatiotemporal correlations in the Takayasu model and in the q model of force fluctuations in bead packs

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We calculate exactly the two point mass-mass correlations in arbitrary spatial dimensions in the aggregation model of Takayasu. In this model, masses diffuse on a lattice, coalesce upon contact, and adsorb unit mass from outside at a constant rate. Our exact calculation of the variance of mass at a given site proves explicitly, without making any assumption of scaling, that the upper critical dimension of the model is 2. We also extend our method to calculate the spatiotemporal correlations in a generalized class of models with aggregation, fragmentation, and injection, which include, in particular, the q model of force fluctuations in bead packs. We present explicit expressions for the spatiotemporal force-force correlation function in the q model. These can be used to test the applicability of the q model in experiments.

I. INTRODUCTION

It is well established that interacting many-body nonequilibrium systems, evolving via the dynamics of their microscopic degrees of freedom, can reach a large variety of steady states in the limit of large time. Among these steady states, those characterized by power-law distributions of different physical quantities, over a wide range of parameter space, have received a lot of attention in the last decade. The phenomenon of the emergence of power-law distributed steady states without fine tuning of external parameters has been dubbed self-organized criticality (SOC) [1]. The concept of SOC has been very useful in understanding scale invariance in a large number of physical systems including sand piles, driven interfaces, river networks, and earthquakes. One of the early models of SOC is the mass aggregation model proposed by Takayasu [2]. This simple lattice model describes a system in which masses diffuse, coalesce upon contact, and adsorb unit mass from outside at a constant rate. In the limit of large time, the mass distribution evolves into a time-independent form with a power-law tail for large mass, in all dimensions. This time-independent mass distribution was computed exactly in one dimension and within mean-field theory [2]. The appearance of a power-law distribution without fine tuning of parameters and the fact that the power law exponent can be computed exactly makes the Takayasu model (TM) one of the simplest analytically tractable models of SOC.

TM also has close connections [3] to other models of statistical mechanics such as the Scheidegger river network model [4], the directed Abelian sandpile model [5], and the voter model [6]. TM is exactly equivalent to Scheidegger's river model, which had been proposed to explain theoretically the observed drainage patterns of river catchment area, in particular Hack's law that relates the catchment area to the length of the river [7]. Though more sophisticated models have been proposed to explain Hack's law, Scheidegger's model is a simple stochastic model that explains the law with fair accuracy. In the case of the directed Abelian sandpile

model the probability of having an avalanche of size m turns out to be identical to the mass distribution $P(m)$ in the TM. The voter model space-time trajectory is very similar to that of the TM except that the processes proceed in the opposite direction in time. Thus, a complete solution of the TM would also help in understanding these related models better.

There has been a recent revival of interest in the TM as simple modifications of the model led to the understanding of a variety of systems including force fluctuations in granular media such as bead packs [8], various aspects of river networks [9], particle systems in one dimension with long-range hops [10], and generalized mass transport models [11]. In addition, two other natural generalizations of the TM were found to display nontrivial nonequilibrium phase transitions in the steady state [12,13].

Even though the single-site mass distribution function in the TM has been computed exactly in one-dimension and in the mean-field limit, the spatial and temporal correlations between masses have remained an open question. In fact, Takayasu and Takayasu, in their recent review article [14], have commented on the difficulty of computing the spatial mass-mass correlation function both analytically and numerically. In this paper, we calculate exactly both the spatial and temporal mass-mass correlation functions in the TM in all dimensions. The calculation of the variance of mass in all dimensions also settles rigorously that the upper critical dimension of the TM, beyond which the mean-field exponents are correct, is 2.

Our technique can also be used to calculate the spatiotemporal two-point correlations in a class of models that are generalizations of the TM. In particular we calculate exactly the two-point force-force correlation function in the q model of force fluctuations in bead packs [8,15]. If experiments can be devised to measure these spatiotemporal correlations in bead packs, then our exact results would be useful for comparison.

The paper is organized as follows. In Sec. II, we define the TM and briefly review the earlier results on this model. We also review the known results for the q model of force fluctuations in bead packs and then summarize our main re-

sults. In Sec. III, we study the spatiotemporal correlations in the TM in one dimension. In Sec. III A, we compute exactly the equal-time mass-mass correlations between two points in space for all times. We also perform a Monte Carlo simulation to compute this correlation function numerically and find excellent agreement between numerical and exact results. In Sec. III B, we solve exactly the temporal autocorrelation and derive its large time behavior. In Sec. IV, we study the spatiotemporal mass-mass correlations in arbitrary dimensions. In Sec. V, we study exactly the spatiotemporal correlations in the generalized q model. We present the explicit results for the correlation functions in the q-model when the distribution of transmitted fractions of weights are uniform. We also provide numerical evidence for the equal-time correlations. Finally, we conclude in Sec. VI with a summary and outlook.

II. MODEL AND RESULTS

A. Takayasu model

For simplicity we define the TM on a one-dimensional lattice with periodic boundary conditions. Generalization to higher dimensions is straightforward. Each site of the lattice has a nonnegative mass variable. Given a certain configuration of masses at time t, the system evolves via the following dynamics. Evolution at each discrete time step consists of two moves: (1) with probability 1/2 each mass hops to its right and with probability 1/2 it stays at the original site and (2) a unit mass is added to each site. The first move corresponds to the diffusion of masses while the second move corresponds to injection of unit masses from outside. If the diffusion move (1) results in two masses coming to the same site, then the total mass at the site simply adds up. Thus, the evolution of the masses is described by the stochastic equation,

$$m_i(t+1) = m_i(t)(1-r_i) + m_{i-1}(t)r_{i-1} + 1, \qquad (1)$$

where the r_i's are independent and identically distributed random variables taking values 0 or 1 with equal probability 1/2, and m_i is the mass at site i. While the first move (1) tends to create big masses via diffusion and aggregation, the second move (2) replenishes the lower end of the mass spectrum. The competition between the two, leads in the long time limit, to a time-independent single-site mass distribution with a power-law tail for large mass. This happens irrespective of the initial condition. For convenience, one can start from an initial configuration that has zero mass at each site.

Note that the dynamics of the TM defined above is parallel, i.e., all sites are updated simultaneously in every time step. Alternately one could define the model in continuous time where the mass at every site hops to the right with rate p and injection of unit masses at every site occurs with rate 1. It turns out that the large distance and long time behaviors of the TM are insensitive to the particular type of dynamics used. This is in contrast to other recently studied generalized mass models [10,11] where the steady-state mass distribution depends nonuniversally on the type of dynamics. It turns out that while the parallel version of the TM is convenient for establishing the mapping to other models, the continuous

time version is sometimes easier for the purpose of calculations. We will therefore use either of the two versions, whichever is more convenient in a particular situation. For example, in one dimension we will calculate the mass-mass correlation function with parallel dynamics, while for arbitrary dimensions d, we will use the continuous time version as significant simplifications occur in that case.

We now briefly review the earlier results on the TM. Takayasu and co-workers [2,14] originally showed that the probability $P(m,t)$ that a site has mass m at time t approaches a time-independent form in the long time limit $t \rightarrow \infty$. This time-independent distribution was shown to have a power-law tail, $\sim m^{-\tau}$ for large m and the exponent τ was computed exactly [2] in $d=1$ ($\tau_{1d}=4/3$) and within mean-field theory ($\tau_{mf}=3/2$). It was also shown that for large m and large but finite t, the distribution satisfies a scaling form [16,17]

$$P(m,t) \sim \frac{1}{m^\tau} f\left(\frac{m}{t^\delta}\right), \qquad (2)$$

where the exponent δ is related to τ via the simple scaling relation $\delta = 1/(2-\tau)$ [17]. Recently Swift *et al.* [18] have argued that in d spatial dimensions, $\tau = 2(d+1)/(d+2)$ for $d < 2$ and $\tau = 3/2$ for $d > 2$. In $d=2$, they argued that in the limit $t \rightarrow \infty$, $P(m) \sim m^{-3/2}(\log m)^{1/2}$ for large m indicating that 2 is the upper critical dimension of the TM. It is also known [2] that the power-law distribution of mass is stable with respect to fluctuations in the initial conditions and is insensitive to whether the particles hop symmetrically in space or not.

While the single-site mass distribution in the TM is known analytically as mentioned above, the spatiotemporal correlations between masses are far from being understood. Recently Takayasu and Takayasu have pointed out that while there exist spatial correlations in TM, they are difficult to compute not only analytically but even numerically [14]. In addition there is no rigorous derivation of the upper critical dimension d_c of the model. While equivalence with the voter model would predict $d_c=2$ [5], early numerical simulations [2] suggested $d_c=4$. The argument of Swift *et al.* [18] that $d_c=2$ is also not rigorous as it relied on a scaling ansatz for the age distribution of particles, which looked plausible but was not proved.

B. The q model of force fluctuations

The q model was proposed [15,8] as a simple scalar model to understand the distribution of forces observed in real three-dimensional bead packs [15]. This model assumed that the force chains observed in experiments were due to inhomogeneity in packing leading to unequal distribution of weights supported by a bead. Ignoring the spatial correlations between inhomogeneities, the model considered a regular lattice of sites, each containing a bead of mass unity. The total weight on a given bead at a layer is transmitted randomly to 2 nearby beads in the layer underneath (In the original version of the model, N adjacent beads were considered). Let $m(i,t)$ be the weight supported by a bead at site i at depth t (t is the layer index). Then the transmission of weights can be represented via the stochastic equation, Eq.

(1), where it is assumed, for convenience, that the successive layers in the t direction are shifted by one lattice unit to the right. The injection term one represents the weight of a bead itself (assuming that all beads have the same weight unity) and r_i represents the fraction of the weight that is transmitted from a given bead to its descendent in the next layer to the right. The only difference with the TM is that, in the q model, r_i's are independent and identically distributed random variables drawn from a uniform distribution over $[0,1]$. Indeed one can study a general stochastic equation such as Eq. (1), where r_i's are independent and identically distributed random variables drawn from a general distribution $f(r)$ over $[0,1]$ [8]. TM is a special case with $f(r) = \frac{1}{2} \delta_{r,0} + \frac{1}{2} \delta_{r,1}$. Similarly the q model with uniform distribution is another special case when $f(r) = 1$.

For distributions of the form $f(r) = r^n(1-r)^n/B(n+1,n+1)$ [where n is a positive integer and $B(m,n)$ is the Beta function], Coppersmith *et al.* showed [8] that the joint probability distribution of the normalized weight variables, $v_i = m_i/t$, factors in the limit $t \to \infty$, i.e., $P(v_1,v_2,v_3\dots) = \Pi_i P(v_i)$ as $t \to \infty$. The uniform distribution falls in this category as it corresponds to $n=0$. Thus the correlations between normalized weights vanishes in the $t \to \infty$ limit and the single point weight distribution was shown to be, $P(v) = a(n)v^{n+1}\exp(-2nv)$ for all v, where $a(n)$ is independent of v [8]. Experiments on bead packs measured the force distribution on the bottom layer of the pack ($t \to \infty$) and the results were found to be in agreement with the q-model results with $n=0$, i.e., with uniform distribution, $f(r)=1$ [15].

While the spatial correlations between the normalized weights vanish in the $t \to \infty$ limit, they are expected to be nonzero at finite depth t. We will show later that Claudin *et al.* [19] stated incorrectly that for the uniform distribution, the correlation is zero at any finite depth. Claudin *et al.* also calculated [19] the equal-depth correlation function in a continuum version of the q model for generic distribution of the fractions and found a rather structureless correlation function. This is not surprising because they made the assumption that the beads are massless. Our exact calculation for the discrete q model in this paper shows that for nonzero bead mass, the equal-depth correlation function has a very interesting scaling behavior characterized by a universal scaling function that is independent of initial conditions for short-ranged initial conditions. Besides, we also compute exactly the nontrivial temporal correlations between masses in the vertical direction. These temporal correlations have not been computed for the q model before.

C. New results

The new results that we obtain in this paper can be summarized as follows:

(1) For the TM, we calculate exactly the equal-time mass-mass correlation function $C(r,t) = \langle m_0(t)m_r(t)\rangle - \langle m_0(t)\rangle \langle m_r(t)\rangle$ between two spatial points separated by a distance r in all dimensions. We show that in the scaling limit $r \to \infty$, $t \to \infty$ but keeping $r/\sqrt{t}$ fixed, $C(r,t) = -t^\gamma G(r/\sqrt{t})$, where $\gamma = 2$ for $d < 2$ and $\gamma = (3 - d/2)$ for $d > 2$ and the scaling function depends on d. In $d = 2$, there are additional logarithmic corrections. In $d = 1$, we also compute the scaling function $G(y)$ explicitly.

(2) Putting $r = 0$ limit in the explicit expression for $C(r,t)$ allows us to compute the on-site variance, $\langle m^2(t)\rangle$ exactly. For large t, we find $\langle m^2 \rangle \sim t^{(4+d)/2}$ in $d < 2$, $\langle m^2 \rangle \sim t^3/\log t$ in $d = 2$, and $\langle m^2 \rangle \sim t^3$ in $d \geq 3$. For $d = 1$, this result was already derived by Takayasu *et al.* [2] but for $d \neq 1$ it is a new result. This therefore proves rigorously, without any assumption of scaling, that the upper critical dimension (beyond which mean-field exponents are correct) of the TM is $d_c = 2$.

(3) We also study exactly the normalized unequal time correlation function,

$$A_r(t,\tau) = \langle X_0(t)X_r(t+\tau)\rangle/\sqrt{\langle X_0^2(t)\rangle\langle X_r^2(t+\tau)\rangle},$$

where $X_r(t) = m_r(t) - \langle m_r(t)\rangle$, in all dimensions in the TM. The normalized autocorrelation function is obtained by putting $r = \tau/2$ in $A_r(t,\tau)$. This is because each mass in the TM has a net drift velocity equal to 1/2 to the right. We show that the autocorrelation function $A_{\tau/2}(t,\tau) \sim \tau^{-d/2}h(\tau/t)$ in the scaling limit, $\tau \to \infty$, $t \to \infty$ but keeping τ/t finite. In $d = 1$, we derive the scaling function $h(y)$ explicitly.

(4) We also calculate exactly the correlations between forces at two different points (both equal depth and unequal depth) in the q model [8] of force fluctuations in bead packs for arbitrary distributions of the fractions of weights. These correlations have so far not been measured experimentally. But if experiments can be performed in the future, then our exact results will be useful for validation of the q model.

III. CORRELATIONS IN ONE DIMENSION

In this section, we calculate exactly the spatiotemporal correlation function in the TM in one dimension. Even though the evolution equation for the single point probability distribution of mass $P(m,t)$ involves the joint two-point probability distribution function $P(m_1,m_2,t)$, the evolution equation for the two-point correlation involves only other two point correlation functions. This simplifying aspect makes the correlation function analytically tractable in the TM.

The parallel dynamics of the TM in one dimension (1D) is represented by the stochastic equation (1), namely, $m_i(t+1) = m_i(t)(1-r_i) + m_{i-1}(t)r_{i-1} + 1$. If $r_i = 1$ at time t, the mass m_i at site i jumps to its right neighbor while $r_i = 0$ indicates that it stays at site i. The hopping of the mass at site i to $i+1$ is described by the first term while the second term accounts for the mass at $i-1$ hopping onto site i. The last term 1 indicates the injection of unit mass from outside at every time t. Averaging Eq. (1) over all possible histories (starting from a zero-mass initial configuration) we immediately get $\langle m \rangle(t) = t$.

A. Equal-time correlations in one dimension

The evolution equation for the equal-time correlation function between two space points i and j can be written down by multiplying Eq. (1) by $m_j(t+1)$ and then taking an average over all possible histories. Due to the translational invariance in an infinite lattice, this correlation function depends only on the difference $|i-j|$. Denoting $x = i-j$ and

using the translational invariance, we find the connected part of the correlation function at time t, $C_x(t)=\langle m_0(t)m_x(t)\rangle-t^2$, obeys the equation,

$$C_x(t+1)=\frac{1}{4}(C_{x+1}+2C_x+C_{x-1})$$

$$+\frac{1}{4}(C_0+t^2)(2\delta_{x,0}-\delta_{x,1}-\delta_{x,-1}). \quad (3)$$

In obtaining the above equation, we have used the fact that r_i and m_j at time t are independent of each other for all i and j. Equation (3) can be solved exactly for arbitrary initial condition by the generating function method. It turns out, however, that the solution at large time t becomes asymptotically independent of the initial condition as long as the initial condition is short ranged. Without any loss of generality, we therefore start from the simplest initial condition when the mass is 0 at every site. Let $F(q,u)=\Sigma_{x=1}^{\infty}\Sigma_{t=0}^{\infty}C_x(t)q^xu^t$. Multiplying Eq. (3) by q^xu^t and summing over x and t, one can express $F(q,u)$ in terms of $\tilde{C}_1(u)=\Sigma_{t=0}^{\infty}C_1(t)u^t$ and $\tilde{C}_0(u)=\Sigma_{t=0}^{\infty}C_0(t)u^t$. $\tilde{C}_1(u)$ can further be expressed in terms of $\tilde{C}_0(u)$ from Eq. (3) by putting $x=0$, multiplying by u^t and summing over t. Thus, finally we get the following expression for $F(q,u)$,

$$F(q,u)=\frac{q[u^2(1+u)(1-q)-2(1-u)^4\tilde{C}_0(u)]}{(1-u)^3[4q-u(1+q)^2]}, \quad (4)$$

where $\tilde{C}_0(u)$ is yet to be determined. We determine $\tilde{C}_0(u)$ by noting that $F(q,u)$ has two poles $q_{1,2}=(2-u\pm2\sqrt{1-u})/u$. For positive values of u, $|q_1|>1$ while $|q_2|<1$. This would imply that for fixed time, $C_x(t)$ will blow up exponentially as $|q_2|^{-x}$ for large x. Since this cannot happen, the numerator on the right-hand side of Eq. (4) must also vanish at $q=q_2$ in order to cancel the pole. This immediately determines $\tilde{C}_0(u)$,

$$\tilde{C}_0(u)=\frac{u(1+u)(1-\sqrt{1-u})}{(1-u)^{7/2}}. \quad (5)$$

Substituting $\tilde{C}_0(u)$ in Eq. (4), the generating function $F(q,u)$ is then fully determined,

$$F(q,u)=\frac{-u(1+u)}{(1-u)^3}\sum_{n=1}^{\infty}\left(\frac{uq}{(1+\sqrt{1-u})^2}\right)^n. \quad (6)$$

Let us denote $\tilde{C}_x(u)=\Sigma_{t=0}^{\infty}C_x(t)u^t$. The coefficient of q^x for $x\geq1$ can be easily pulled out from Eq. (6) to yield

$$\tilde{C}_x(u)=\frac{-u(1+u)}{(1-u)^3}\frac{u^x}{(1+\sqrt{1-u})^{2x}}. \quad (7)$$

Note that it is evident from the above expression that $C_x(t)=0$ for $x\geq t$.

In order to derive the explicit expressions for $C_x(t)$ for $x\geq0$, we need to invert the discrete Laplace transforms

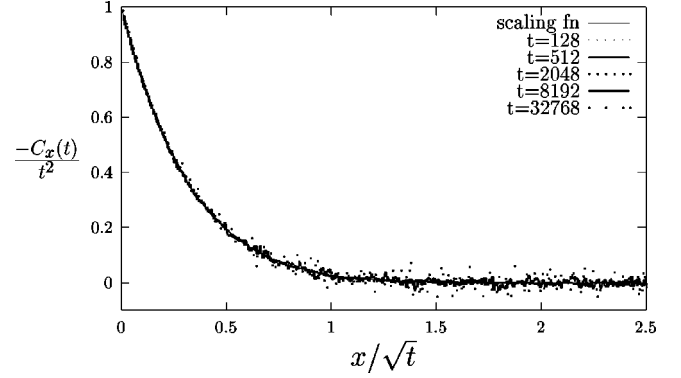

FIG. 1. The figure shows the scaling plots of the equal-time correlation function, $C_x(t)=\langle m(0,t)m(x,t)\rangle-t^2$ obtained from numerical simulation of the TM on a one-dimensional lattice of 1000 sites. The data for five different times collapse onto a single scaling curve that matches very well with the analytical scaling function given by Eq. (12).

$\tilde{C}_x(u)$. We first derive $C_0(t)$ explicitly by computing the coefficient of u^t in the expression of $\tilde{C}_0(u)$ in Eq. (5),

$$C_0(t)=\frac{2t(2t+1)(4t+1)}{4^t15}\frac{(2t)!}{(t!)^2}-t^2. \quad (8)$$

This therefore gives us an exact expression for the on-site mass variance $C_0(t)$ for all t. Taking the large t limit in Eq. (8) we get,

$$C_0(t)=\frac{16t^{5/2}}{15\sqrt{\pi}}+O(t^2). \quad (9)$$

We note that since Eq. (2) implies that $\langle m^2\rangle\sim t^{(3-\tau)/(2-\tau)}$, we get $\tau=4/3$ in one dimension. This therefore constitutes an alternate method to derive $\tau=4/3$ in one dimension.

In order to derive $C_x(t)$ for $x>0$, we need to calculate the coefficient of u^t on the right-hand side of Eq. (7). For arbitrary x, this is somewhat hard. However it is easy to derive the asymptotic behavior of $C_x(t)$ for large x and large t but keeping $x/\sqrt{t}$ fixed. By taking $u\rightarrow1$ limit in Eq. (7) and after a few steps of algebra, we find that in this scaling limit

$$C_x(t)=-t^2G(x/\sqrt{t}), \quad (10)$$

where the scaling function $G(u)$ is universal, i.e., independent of the initial condition as long as the initial condition is short ranged and is given by

$$G(y)=32\int_y^{\infty}dx_1\int_{x_1}^{\infty}dx_2\int_{x_2}^{\infty}dx_3\int_{x_3}^{\infty}dx_4\ \mathrm{erfc}(x_4). \quad (11)$$

The complementary error function is defined as, $\mathrm{erfc}(y)=(2/\sqrt{\pi})\int_y^{\infty}\exp(-x^2)dx$. The above integrals can be done to derive an explicit expression for the scaling function,

$$G(y) = \frac{1}{3}\left[(3+12y^2+4y^4)\mathrm{erfc}(y) - \frac{2}{\sqrt{\pi}}y(5+2y^2)e^{-y^2} \right].$$

(12)

We also performed a numerical simulation of the TM on a one-dimensional lattice with periodic boundary condition. In Fig. 1, we show the scaling plot of the connected part of the correlation function. The data at different times, when scaled as in Eq. (10) collapse onto a single scaling function, which is in excellent agreement with the analytical result given by Eq. (12).

B. Temporal correlations in one dimension

In this section we compute exactly the two-time correlations in the TM in one dimension. Let us first define $X_x(t) = m_x(t) - \langle m_x(t) \rangle = m_x(t) - t$. We then define the general two-time correlation function as $D_x(t,\tau) = \langle X_0(t)X_x(t+\tau) \rangle$. It is also useful to define the normalized two-time correlation function,

$$A_x(t,\tau) = \frac{\langle X_0(t)X_x(t+\tau)\rangle}{\sqrt{\langle X_0^2(t)\rangle\langle X_x^2(t+\tau)\rangle}}.$$

(13)

Clearly then, $A_x(t,\tau) = D_x(t,\tau)/\sqrt{C_0(t)C_0(t+\tau)}$ where $C_0(t)$ is just the on-site mass variance whose exact expression is given by Eq. (8) of the previous section and we have also used the translational invariance of the equal-time correlation function. Thus we just need to evaluate $D_x(t,\tau)$ which can be done exactly as follows.

From Eq. (1), it is easy to show that the function $D_x(t,\tau)$ evolves as a function of τ for fixed t as,

$$D_x(t,\tau+1) = \tfrac{1}{2}[D_x(t,\tau) + D_{x-1}(t,\tau)],$$

(14)

starting from the initial condition $D_x(t,0) = C_x(t)$, where $C_x(t)$ is the equal-time correlation function computed already in the previous section. Let $H(k,t,\tau) = \Sigma_{x=-\infty}^{\infty}D_x(t,\tau)e^{ikx}$. Note that here we used the x summation from $-\infty$ to ∞ as opposed to 0 to ∞. This is because $D_x(t,\tau)$ is not equal to $D_{-x}(t,\tau)$ for $\tau > 0$. They become equal only for $\tau = 0$ due to translational invariance. From Eq. (14) we get

$$H(k,t,\tau) = H(k,t,0)\left(\frac{1+e^{ik}}{2}\right)^\tau.$$

(15)

Note that $H(k,t,0) = \Sigma_{x=-\infty}^{\infty}C_x(t)e^{ikx}$ and $C_{-x}(t) = C_x(t)$ as translational invariance holds for equal-time correlation functions.

By inverting the Fourier transform in Eq. (15), we get a simple expression for $D_x(t,\tau)$ in terms of the equal-time correlation functions,

$$D_x(t,\tau) = \frac{1}{2\pi}\int_0^{2\pi} dk H(k,t,\tau)e^{-ikx}$$

$$= \frac{1}{2^\tau}\sum_{m=0}^{\tau}\binom{\tau}{m}C_{x-m}(t).$$

(16)

In order to calculate the autocorrelation function, we note that in the TM, the masses have a net drift velocity, $v = 1/2$ towards the right. This is because of the definition of the model: in one time step, a mass either stays at its own site with probability 1/2, or hops to the neighbor on the right with probability 1/2. Thus, to calculate the proper autocorrelation function, one has to compute it in the moving frame that is shifting towards the right with uniform velocity 1/2. Hence the correct auto-correlation function would be given by $D_{\tau/2}(t,\tau)$. Putting $x = \tau/2$ in Eq. (16) and taking the transform $\tilde{D}_{\tau/2}(u,\tau) = \Sigma_{t=0}^{\infty}D_{\tau/2}(t,\tau)u^t$, we get

$$\tilde{D}_{\tau/2}(u,\tau) = \frac{1}{2^\tau}\left[\binom{\tau}{\tau/2}\tilde{C}_0(u) + 2\sum_{m=\tau/2+1}^{\tau}\binom{\tau}{m}\tilde{C}_{m-\tau/2}(u)\right],$$

(17)

where we have used the symmetry $\tilde{C}_x(u) = \tilde{C}_{-x}(u)$. Using the exact expressions for $\tilde{C}_x(u)$ from Eqs. (5) and (7) of the previous section in Eq. (17), and after some steps of algebra, we get

$$\tilde{D}_{\tau/2}(u,\tau) = \frac{1}{2^\tau}\binom{\tau}{\tau/2}\frac{u(1+u)}{(1-u)^{7/2}} - \frac{u(1+u)}{(1-u)^3u^{\tau/2}}$$

$$+ \frac{u(1+u)}{(1-u)^{5/2}u^{\tau/2}}\sum_{i=0}^{\tau/2-1}\binom{2i}{i}\left(\frac{u}{4}\right)^i,$$

(18)

where we have used the combinatorial identity [20],

$$\sum_{j=n/2+1}^{n}\binom{n}{j}k^j = \frac{(1+k)^n}{2} - \frac{k^{n/2}}{2}\binom{n}{n/2} - \frac{(1-k)(1-k)^{n-1}}{2}$$

$$\times \sum_{i=0}^{n/2-1}\binom{2i}{i}\left(\frac{k}{(1+k)^2}\right)^i.$$

(19)

In order to analyze Eq. (18), we first put $u = 1-s$ and note that the equation allows a scaling limit when $s \to 0$ and $\tau \to \infty$, but the product $s\tau$ remains fixed. In terms of time variables, this scaling limit corresponds to $\tau \to \infty$, $t \to \infty$ but keeping the ratio τ/t fixed. In this limit, we find

$$\tilde{D}_{\tau/2}(s,\tau) = \frac{1}{s^3}g_1(s\tau),$$

(20)

where the scaling function is given by

$$g_1(y) = \sqrt{\frac{8}{\pi}}\left[\frac{1}{\sqrt{y}} - \sqrt{\frac{\pi}{2}}e^{y/2}\,\mathrm{erfc}\left(\sqrt{\frac{y}{2}}\right)\right].$$

(21)

In terms of time variables τ and t in the scaling limit, $\tau \to \infty$ and $t \to \infty$, but keeping τ/t fixed, we get the following expression by inverting the Laplace transform in Eq. (20),

$$D_{\tau/2}(t,\tau) = t^2\left[16\sqrt{\frac{2}{15\pi}}\sqrt{\frac{t}{\tau}} - h_1\left(\frac{\tau}{t}\right)\right],$$

(22)

where

$$h_1(y) = \frac{(1+y/2)^2}{3\pi}\left[6\sin^{-1}\left(\frac{1}{\sqrt{1+y/2}}\right) - \frac{\sqrt{y}(10+3y)}{\sqrt{2}(1+y/2)^2}\right].$$

(23)

Using the above result and the exact large t behavior of $C_0(t)$ from Eq. (9) in Eq. (13), we finally obtain the scaling behavior of the normalized autocorrelation function in the scaling limit mentioned above,

$$A_{\tau/2}(t,\tau) = \frac{1}{\sqrt{\tau}}h\left(\frac{\tau}{t}\right),$$

(24)

where the scaling function is given by,

$$h(y) = \sqrt{\frac{2}{\pi}}\frac{1}{(1+y)^{5/4}}\left\{1 - \frac{5\sqrt{2y}}{32}(1+y/2)^2\right.$$
$$\left. \times\left[6\sin^{-1}\left(\frac{1}{\sqrt{1+y/2}}\right) - \frac{\sqrt{y}(10+3y)}{\sqrt{2}(1+y/2)^2}\right]\right\}.$$

(25)

The function $h(y) \to \sqrt{2/\pi}$ as $y \to 0$ and $h(y) \approx \sqrt{(8/49\pi}y^{-9/4}$ as $y \to \infty$. Thus, for large y, the scaling function decays as a power law with an exponent 9/4.

We remark that the above scaling behavior holds only in the limit when $\tau \to \infty$, $t \to \infty$ but the ratio τ/t is kept fixed. In other limits, it is also possible to investigate the detailed behavior of the autocorrelation function by analyzing the exact equation (18).

IV. CORRELATIONS IN ARBITRARY DIMENSIONS

In this section we study the two-point spatiotemporal correlations in the TM in an arbitrary spatial dimension d. As mentioned in Sec. II, it turns out that for general d, equations for the correlations simplify considerably for the continuous time version of the TM. In this version, every mass hops with rate p to each of its d nearest neighbors in the positive direction, and aggregates with the mass present at the hopped site. In addition, injection of unit mass occurs at every lattice site with rate 1.

The evolution of the mass $m(x_1, x_2, \ldots, x_d, t)$ in a small time interval Δt can be represented by the equation,

$$m(\{x_i\}, t+\Delta t) = \sum_{j=1}^{d} r_j^- m(x_1, \ldots, x_j-1, \ldots, x_d, t)$$
$$+ \left(1 - \sum_{j=1}^{d} r_j^+\right)m(\{x_i\}, t) + I(\{x_i\}, t),$$

(26)

where $r_i^{\pm}$'s are independent and identically distributed variables, with distribution $f(r) = p\Delta t \delta_{r,1} + (1-p\Delta t)\delta_{r,0}$ and indicate the hopping events of the particles. The random variable $I(\{x_i\}, t)$ denotes the event of injection and is drawn from the distribution $P(I) = \Delta t \delta_{I,1} + (1-\Delta t)\delta_{I,0}$ independent of the $r_i^{\pm}$'s.

A. Equal-time correlations

From the Eq. (26) of evolution of the masses, one can easily write down the evolution equation for the two-point correlation function in continuous time. Multiplying Eq. (26) at two different space points and neglecting terms of order $O(\Delta t^2)$ and higher, we find that the two-point correlation function $C(\{x\}, t) = \langle m(0, \ldots, 0, t)m(x_1, \ldots, x_d, t)\rangle - t^2$, evolves as

$$\frac{d}{dt}C(\{x\}) = -2pdC(\{x\}) + p\sum_{j=1}^{d}\sum_{m=\pm 1}C(x_1, \ldots, x_j$$
$$+m, \ldots, x_d) + \delta_{x_1,0}\cdots\delta_{x_d,0} + p(C(\{0\})+t^2)$$
$$\times\left(2d\delta_{x_1,0}\cdots\delta_{x_d,0}\right.$$
$$\left. -\sum_{j=1}^{d}\delta_{x_1,0}\cdots\delta_{x_j,\pm 1}\cdots\delta_{x_d,0}\right),$$

(27)

where we have suppressed the t dependence of $C(\{x\}, t)$ for notational convenience and also used a translational invariance of $C(\{x\}, t)$. The Fourier transform $G(\{k\}, t) = \sum_{\{x\}=-\infty}^{\infty}C(\{x\})\exp(i\sum_{j=1}^{d}k_jx_j)$ then evolves as,

$$\frac{d}{dt}G(\{k\}) = 2p[G(\{k\}) - C(\{0\}) - t^2]$$
$$\times\left(-d+\sum_{j=i}^{d}\cos(k_j)\right)+1.$$

(28)

Taking the Laplace transform with respect to t, we get

$$F(\{k\}, s) = \frac{s^2 + 2p[s^3g_0(s)+2]\left[d - \sum_{j=1}^{d}\cos(k_j)\right]}{s^3\left\{s + 2p\left[d - \sum_{j=1}^{d}\cos(k_j)\right]\right\}},$$

(29)

where $F(\{k\}, s) = \int_0^{\infty}G(\{k\}, t)e^{-st}dt$ and $g_0(s) = \int_0^{\infty}C(0, \ldots, 0, t)e^{-st}dt$, which is yet to be determined. We determine $g_0(s)$ by noting that $g_0(s) = 1/(2\pi)^d\int_0^{2\pi}\cdots\int_0^{2\pi}F(\{k\}, s)dk_1\cdots dk_d$. Integrating Eq. (29) with respect to the k_i's we get the following expression for $g_0(s)$,

$$g_0(s) = \frac{1}{s^4}\left(s^2 - 2s + \frac{2}{I(s)}\right),$$

(30)

where

$$I(s) = \frac{1}{(2\pi)^d}\int_0^{2\pi}dk_1\cdots\int_0^{2\pi}dk_d\frac{1}{s + 2p\left[d - \sum_{j=1}^{d}\cos(k_j)\right]}.$$

(31)

The small s behavior of $I(s)$ can be easily evaluated by analyzing the integral in Eq. (31). We find that as $s \to 0$,

$$I(s) \sim s^{-(1-d/2)} \quad \text{for} \quad d<2,$$

$$\sim -\log(s) \quad \text{for} \quad d=2,$$

$$\sim \text{const} \quad \text{for} \quad d>2. \tag{32}$$

Substituting $I(s)$ in the expression for $g_0(s)$ in Eq. (30) and inverting the Laplace transform, we find that the on-site variance for large t behaves exactly as,

$$C(0,\ldots,0,t) \sim t^{4+d/2} \quad \text{for} \quad d<2,$$

$$\sim \frac{t^3}{\log(t)} \quad \text{for} \quad d=2,$$

$$\sim t^3 \quad \text{for} \quad d>2. \tag{33}$$

Note that the results in Eq. (33) are exact results for large t and does not assume any scaling behavior. This result clearly proves *rigorously* that the upper critical dimension of TM is $d_c = 2$.

In addition, if we assume that the on-site mass distribution scales as $P(m,t) \sim m^{-\tau} f(mt^{-\delta})$ for large m and large t, we get the following results for τ and δ. The first moment $\langle m \rangle \sim t$ gives $\delta = 1/(2-\tau)$ [17]. The second moment scales as $\langle m^2 \rangle \sim t^{(3-\tau)\delta}$. Using the exact results for variance from Eq. (33), we get $\tau = 2(d+1)/(d+2)$ for $d<2$, $\tau=3/2$ for $d>2$, and $\tau=3/2$ in $d=2$ with additional logarithmic corrections. These results for τ are in agreement with the results obtained by Swift *et al.* [18] by using a more indirect mapping to the age distribution of particles in a related reaction-diffusion process and also assuming scaling behavior.

With $g_0(s)$ completely determined, we therefore have an exact expression in Eq. (29) for $F(\{k\},s)$, the joint Laplace-Fourier transform of the full correlation function $C(\{x\},t)$ in arbitrary dimensions. For arbitrary d, it is complicated to invert this transform to obtain an exact expression for $C(\{x\},t)$. However, by analyzing the small s and small k behavior of $F(\{k\},s)$, it is easy to see that for large x and large t, but keeping $xt^{-1/2}$ fixed, $C(\{x\},t)$ satisfies a scaling behavior, $C(\{x\},t) \sim t^\gamma G(\{x_i/\sqrt{t}\})$ with $\gamma=2$ for $d=1$ and $\gamma=(3-d/2)$ for $d>2$ and the scaling function $G(y)$ depends explicitly on d.

For $d=2$, there is additional logarithmic correction and the scaling breaks down. In this case, the exact expression for $I(s)$ is given by

$$I = \frac{2}{\pi(s+4p)} K[4p/(s+4p)], \tag{34}$$

where K is a complete elliptic integral [20]. This gives an explicit expression for $g_0(s)$,

$$g_0(s) = \frac{\pi(s+4p)}{s^4 K[4p/(s+4p)]} + \frac{1}{s^2} - \frac{2}{s^3}. \tag{35}$$

Substituting the expression for $g_0(s)$ in Eq. (29) we get,

$$F(k_1,k_2,s)$$

$$= \frac{2}{s^3} - \frac{\pi(s+4p)}{s^3 K[4p/(s+4p)]\{s+2p[2-\cos(k_1)-\cos(k_2)]\}}. \tag{36}$$

After some straightforward algebra, it turns out that the large distance behavior of $\tilde{C}(x,y,s) = \int_0^\infty C(x,y,t)e^{-st}dt$ is given by,

$$\tilde{C}(x,y,s) = \frac{-(s+4p)}{2s^3 pK[4p/(s+4p)]} K_0\left(\frac{r\sqrt{s}}{p}\right), \tag{37}$$

where $r=\sqrt{x^2+y^2}$ and K_0 is the modified Bessel function [20]. In order to get an explicit expression of $C(x,y,t)$ for large r and t, one needs to invert the Laplace transform given by Eq. (37). But it is obvious from this expression that $C(x,y,t)$ will no longer have a nice scaling form in the large distance and long-time limit as in one dimension [see Eq. (10)] due to the appearance of logarithms in the asymptotic behavior of the functions $K(x)$ and $K_0(x)$. This violation of scaling due to logarithms is again expected since two is the upper critical dimension of the TM.

B. Temporal correlations

We can write down the equations for the time evolution of the temporal correlation function in a manner very similar to that in $d=1$. We define the connected correlation function as $D_{x_1,\ldots,x_d}(t,\tau) = \langle m_{0,\ldots,0}(t)m_{x_1,\ldots,x_d}(t+\tau)\rangle - t(t+\tau)$. From the evolution equation of the masses, it is easy to show that the function $D_x(t,\tau)$ evolves as a function of τ for fixed t as,

$$\frac{d}{d\tau}D_{\{x\}}(t,\tau) = p\left(\sum_{j=1}^d (D_{x_1,\ldots,x_j-1,\ldots,x_d}(t,\tau)\right.$$

$$\left. - dD_{\{x\}}(t,\tau)\right), \tag{38}$$

starting from the initial condition $D_{\{x\}}(t,0)=C_{\{x\}}(t)$ where $C_{\{x\}}(t)$ is the equal-time correlation function. Let $H(\{k\},t,\tau) = \sum_{\{x\}=-\infty}^{+\infty} D_{\{x\}}(t,\tau)\exp(\sum_{j=0}^d ik_j x_j)$. From Eq. (38) we immediately get

$$H(\{k\},t,\tau) = H(\{k\},t,0)\exp\left(p\tau\sum_{j=1}^d (e^{ik_j}-1)\right), \tag{39}$$

where $H(\{k\},t,0)$ is the Fourier transform of the equal-time correlation function.

We invert Eq. (39) to get the temporal correlations in terms of the equal-time correlation functions,

$$D_{\{x\}}(t,\tau) = \frac{1}{(2\pi)^d} \int_0^{2\pi} d^d k H(\{k\},t,\tau)\exp\left(-i\sum_{j=1}^d k_j x_j\right),$$

$$= \frac{e^{-p\tau d}}{(2\pi)^d} \int_0^{2\pi} d^d k \exp\left(-i\sum_{j=1}^d k_j x_j\right)$$

$$\times \sum_{\{x'\}=-\infty}^\infty C(\{x'\},t)\exp\left(i\sum_{j=1}^d k_j x_j'\right)$$

$$\times \exp\left(p\tau \sum_{j=1}^d e^{ik_j}\right). \tag{40}$$

As in $d=1$, there is a net drift velocity, $p\tau$, in each forward direction. Hence the correct autocorrelation function is given by $D_{\{p\tau\}}(t,\tau)$. Putting $x_j = p\tau$ in Eq. (40), and simplifying, we get

$$D_{\{p\tau\}}(t,\tau) = \sum_{\{m\}=0}^\infty C(p\tau-m_1,\ldots,p\tau-m_d)$$

$$\times e^{-dp\tau} \frac{(p\tau)^{\sum_j m_j}}{\prod_j m_j!}. \tag{41}$$

It can then be shown that in the scaling limit, $\tau\to\infty$, $t\to\infty$ but keeping τ/t fixed, the normalized autocorrelation function allows for a scaling solution as in $d=1$,

$$A_{\{p\tau\}}(t,\tau) \sim \frac{1}{\tau^{d/2}} h\left(\frac{\tau}{t}\right). \tag{42}$$

We do not present the explicit form of the scaling function here. It can be shown that $h(y)\to$const. as $y\to0$. Thus in arbitrary dimensions, the asymptotic decay of the normalized auto-correlation function is given by $\tau^{-d/2}$ for large τ (after taking the large t limit).

V. CORRELATIONS IN THE q MODEL OF FORCE FLUCTUATIONS

The q model of force fluctuations has been defined in Sec. II B. In this model, the variables m_i evolve in time via the stochastic equation, $m_i(t+1) = (1-r_i)m_i(t) + r_{i-1}m_{i-1}(t) + 1$, where the random variables r_i's are drawn independently from a arbitrary distribution $f(r)$ over the support [0,1]. Experimental results for the force distribution in real bead packs were found to be described accurately by the q model with uniform distribution $f(r)=1$ [15].

In this section, we calculate exactly the two-point correlations between m_i's for the generalized q model, i.e., for any arbitrary distribution $f(r)$. It turns out that the two-point correlations are characterized by two parameters $\mu_1 = \int_0^1 rf(r)dr$ and $\mu_2 = \int_0^1 r^2 f(r)dr$ with $\mu_1 \geq \mu_2$. In the $\mu_1 \geq \mu_2$ plane, there are two types of asymptotic behaviors depending on whether $\mu_1 = \mu_2$ or $\mu_1 > \mu_2$. At every point on the line $\mu_1 = \mu_2$, the correlation function has the same universal asymptotic behavior independent of initial conditions as long as they are short ranged. The special case of the TM with $\mu_1 = \mu_2 = 1/2$ falls within this class. On the other hand,

for all points in the plane $\mu_1 > \mu_2$, the correlation function has once again the same universal asymptotic behavior regardless of the actual values of μ_1 and μ_2 but this behavior is different from the behavior on the line $\mu_1 = \mu_2$. The q model with uniform distribution corresponds to the point $\mu_1 = 1/2$, $\mu_2 = 1/3$ and therefore falls in the second category.

Since the steps of the calculation follow closely that of the TM, we will skip most of the details and present only the final results.

A. Equal-time correlations in one dimension

Starting from the stochastic evolution equation [Eq. (1)] of masses in $d=1$, it is straightforward to write down the evolution equation for the equal-time correlation function $C_x(t) = \langle m_0(t)m_x(t)\rangle - t^2$. We find that for general distribution $f(r)$, the equal-time correlations evolve as

$$C_x(t+1) = (\mu_1 - \mu_1^2)(C_{x+1} + C_{x-1}) + (1 - 2\mu_1 + 2\mu_1^2)C_x$$

$$+ (\mu_2 - \mu_1^2)(C_0 + t^2)(2\delta_{x,0} - \delta_{x,1} - \delta_{x,-1}), \tag{43}$$

where $\mu_1 = \int_0^1 rf(r)dr$ and $\mu_2 = \int_0^1 r^2 f(r)dr$. Note that for the TM, $\mu_1 = 1/2$ and $\mu_2 = 1/2$ and then Eq. (43) reduces exactly to Eq. (3) studied in Sec. III A. For the uniform distribution, $f(r)=1$, one gets, $\mu_1 = 1/2$ and $\mu_2 = 1/3$.

We can solve Eq. (43) for arbitrary initial condition and arbitrary parameters μ_1 and μ_2. It turns out that the dependence on the initial condition drops out for asymptotically large t. Following the same steps as in the TM in Sec. III A, we find that the Laplace transforms $\tilde{C}_x(u) = \sum_{t=0}^\infty C_x(t)u^t$, are given exactly by

$$\tilde{C}_0(u) = \frac{u(1+u)[\sqrt{1+(4a-1)u} - \sqrt{1-u}]}{(1-u)^3 g_1(u)}, \tag{44}$$

$$\tilde{C}_x(u) = \frac{-u(1+u)}{(1-u)^{5/2} g_1(u)}\left(\frac{\sqrt{1+(4a-1)u} - \sqrt{1-u}}{\sqrt{4au}}\right)^{2x}, \tag{45}$$

where $g_1(u) = [(a-b)/b\sqrt{1+(4a-1)u} + \sqrt{1-u}]$ and $a = \mu_1 - \mu_1^2$ and $b = \mu_2 - \mu_1^2$. It is clear from the above expressions that the asymptotic behaviors for large t (corresponding to $u\to1$) depend on whether $a=b$, i.e., $\mu_1 = \mu_2$ or $a>b$, i.e., $\mu_1 > \mu_2$.

For $\mu_1 = \mu_2$, we find, after inverting the Laplace transforms, that for large t,

$$C_0(t) \approx \frac{32\sqrt{a}}{15\sqrt{\pi}} t^{5/2}, \quad t \gg 1, \tag{46}$$

$$C_x(t) \approx -t^2 G\left(\frac{x}{\sqrt{4at}}\right), \quad x,t \gg 1, \tag{47}$$

where the universal scaling function $G(y)$ is given by Eq. (12) as calculated for the TM.

For $\mu_1 > \mu_2$, on the other hand, we find for large t,
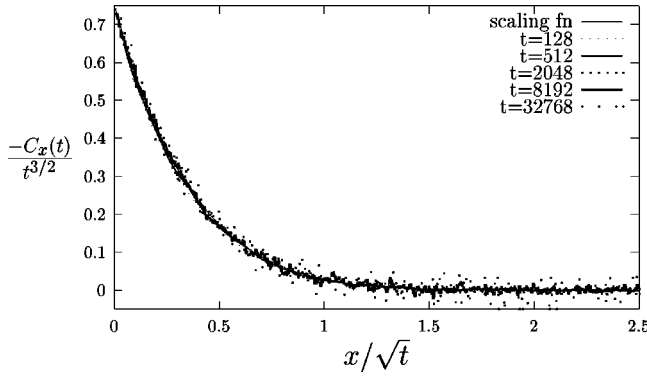
FIG. 2. The figure shows the scaling plots of the equal-time correlation function $C_x(t) = \langle m(0,t)m(x,t)\rangle - t^2$ obtained from numerical simulation of the q model on a one-dimensional lattice of 1000 sites. The data for five different times collapse onto a single scaling curve that matches very well with the analytical scaling function given by Eq. (50).

$$C_0(t) \approx \frac{b}{a-b}t^2, \quad t \gg 1, \tag{48}$$

$$C_x(t) \approx -\frac{b}{\sqrt{a}(a-b)}t^{3/2}G_1\left(\frac{x}{\sqrt{4at}}\right), \quad x,t \gg 1, \tag{49}$$

where the universal scaling function $G_1(y)$ is given by

$$G_1(y) = \frac{2}{3}\left(\frac{2e^{-y^2}(1+y^2)}{\sqrt{\pi}} - y(3+2y^2)\text{erfc}(y)\right). \tag{50}$$

Note that the uniform distribution corresponds to $\mu_1 = 1/2$ and $\mu_2 = 1/3$, i.e., $a = 1/4$ and $b = 1/12$. In Fig. 2, we compare the numerically obtained scaling plots for the uniform distribution with the exact scaling function given by Eq. (50).

If we define the scaled weight $v = m/t$ as in Ref. [8], then the connected part of the two-point correlations $\langle v(0,t)v(x,t)\rangle - 1 \simeq -t^{-1/2}G_1(x/\sqrt{4at})$ for large x and t. Clearly as $t \to \infty$, the scaled weight v's get completely uncorrelated but for finite depth (or time) t, there is a nonzero anticorrelation specified by the scaling function $G_1(u)$ which might be possible to measure experimentally. We also point out that the statement of Claudin *et al.*, [19] that for the uniform case the correlation is zero at any altitude t, is clearly incorrect.

We note that from Eq. (43), one can easily derive the evolution equation for the correlation function in the continuum space and time. For a proper continuum limit in time, we need to assume that both $a \to a\Delta t$ and $b \to b\Delta t$ are of order Δt. Defining the Fourier transform $G(k,t) = \int_{-\infty}^{\infty}C(x,t)e^{ikx}dx$, one finds from Eq. (43) that for small k, the correlation function evolves via the equation,

$$(\partial_t + ak^2)G(k,t) = bk^2\left(\int\frac{dk'}{2\pi}G(k',t) + t^2\right). \tag{51}$$

Note that this equation above is identical to the one derived by Claudin *et al.* [19] except for the additional k^2t^2 term on the right-hand side of Eq. (51). The origin of this additional term can be traced back to the fact that in our case, the bead

mass is nonzero (equal to 1) as opposed to the zero-mass case considered in Ref. [19]. As seen from our analysis, for nonzero mass, the correlation function has a much more nontrivial and universal structure as opposed to the rather structureless correlations found in the zero-mass case in Ref. [19].

Rescaling time by the factor a, one sees that Eq. (51) is parameterized by the single variable $\lambda = b/a$. As in the discrete case, there are two possible asymptotic behaviors depending on the value of λ. For $\lambda = 1$, one finds the Takayasu type behavior and a completely different asymptotic behavior emerges for all $\lambda < 1$.

B. Temporal correlations in one dimension

The two-time correlations can similarly be computed for the q model for any arbitrary distribution $f(r)$. We present here only the explicit results for the uniform distribution in $d = 1$.

For two-time correlations we use the same notation as in the TM [see Sec. III B]. For the uniform distribution $f(r) = 1$, we find that in the scaling limit $\tau \to \infty$ and $t \to \infty$, but keeping τ/t fixed, the normalized autocorrelation function, as defined in Eq. (13), has the scaling behavior

$$A_{\tau/2}(t,\tau) = \frac{1}{\sqrt{\tau}}h_2\left(\frac{\tau}{t}\right), \tag{52}$$

where the scaling function is given by

$$h_2(y) = \sqrt{\frac{2}{\pi}}\frac{1}{3(1+y)}[3 - 2\sqrt{y}(y+2)^{3/2} + 6y + 2y^2]. \tag{53}$$

The scaling function $h_2(y) \to \sqrt{2/\pi}$ as $y \to 0$ as in the TM. For large y also, $h_2(y)$ decays as a power law; $h_2(y) \approx \sqrt{2/9\pi}y^{-2}$ as in the TM but with a different exponent two than for the TM exponent 9/4.

C. Results for arbitrary dimensions

Following the similar line of arguments as in the TM, the evolution equation for the two-point correlation function can be derived for the generalized q model in arbitrary spatial dimension with discrete space and time. The discrete equations are rather complicated but the asymptotic properties can be easily derived by taking continuum space and time limit. In the continuum limit, it turns out that as in $d = 1$, the Fourier transform in any dimension $G(\{k_i\},t) = \int_{-\infty}^{\infty}C(\{x_i\},t)\exp(i\Sigma k_i x_i)d^dx$ evolves by the simple equation (51) parameterized by the ratio $\lambda = b/a$, once time is rescaled by a. The parameter $\lambda \le 1$ as in the discrete case. As in the TM, taking the Laplace transform of Eq. (51) with respect to t, we find

$$F(\{k\},s) = \frac{\lambda k^2[2 + g_0(s)s^3]}{s^3(k^2 + s)}, \tag{54}$$

where $F(\{k\},s) = \int_0^{\infty}G(\{k\},t)e^{-st}dt$ and $g_0(s) = \int_0^{\infty}C(0,\ldots,0,t)e^{-st}dt$, which is yet to be determined. As in the TM, $g_0(s)$ is determined by integrating Eq. (54) with

respect to k. Note that the upper cutoff of each k_i integration is now set by $2\pi/\Lambda$ where Λ is the lattice constant. We find

$$g_0(s) = \frac{2\lambda[1 - sI(s)]}{s^3[1 - \lambda + \lambda sI(s)]}, \qquad (55)$$

where $I(s) = \int d^d k/(k^2 + s)$. The small s behavior of $I(s)$ is the same as given by Eq. (32).

From Eq. (55) it is clear that there are two different small s behaviors of $g_0(s)$ depending on whether $\lambda = 1$ or $\lambda < 1$. The TM corresponds to $\lambda = 1$ and its asymptotic behaviors have already been discussed in detail in Sec. IV. Here we focus on $\lambda < 1$. In that case, using the small s behavior of $I(s)$ in Eq. (55), it is clear that $g_0(s) \sim 2\lambda/s^3$ in any dimension, indicating that $C(0,0,\ldots,0,t) \sim t^2$ for large t. Thus for $\lambda < 1$, contrary to the $\lambda = 1$ case (TM), there is no critical dimension separating different asymptotic growth of $C(0,0,\ldots,0,t)$.

Substituting this expression for $g_0(s)$ in Eq. (54), we find that for small s

$$F(\{k\}, s) \approx \frac{2\lambda k^2}{(1 - \lambda)s^3(k^2 + s)}. \qquad (56)$$

It is then not difficult to derive the asymptotic properties of the correlation function in real space and time. We find,

$$C(\{0\}, t) \sim t^2, \quad t \gg 1, \qquad (57)$$

$$C(\{x\}, t) \approx -t^{(2 - d/2)} G_1\left(\frac{x}{\sqrt{t}}\right), \quad x, t \gg 1, \qquad (58)$$

where $G_1(y)$ is the dimension dependent scaling function.

The most important result of this subsection is that while for $\lambda = 1$, there is an upper critical dimension $d_c = 2$ separating different asymptotic growth of the on-site variance, there is no such critical dimension for $\lambda < 1$ (which includes the uniform distribution of fractions for the q model).

VI. SUMMARY AND CONCLUSION

In this paper we have computed exactly both the equal-time as well as the unequal-time two-point correlations in the Takayasu model of mass aggregation and injection in all dimensions. We have identified different scaling limits and obtained the scaling functions explicitly in $d = 1$. Our exact results for the on-site mass variance prove rigorously, without any assumption of scaling, that the upper critical dimension of the Takayasu model is $d_c = 2$.

We have also extended our technique to compute exactly the correlations in a larger class of aggregation models with injection. This generalized model includes, as special cases, the Takayasu model and also the q model of force fluctuations in granular materials. We have shown that the correlation functions in this generalized model is parameterized by two variables (μ_1, μ_2) which are, respectively, the first and the second moment of the distribution $f(r)$ of the fractions r_i's. We have shown that in the two-dimensional parameter space (μ_1, μ_2) with $\mu_1 \geq \mu_2$, there are two types of asymptotic behaviors of the correlation function depending on whether $\mu_1 = \mu_2$ or $\mu_1 > \mu_2$. For generic points in the region $\mu_1 > \mu_2$, which includes the uniform distribution represented by $(1/2, 1/3)$, the correlations have similar asymptotic behaviors, which is different from that on the line $\mu_1 = \mu_2$, which includes the Takayasu model. Besides, for $\mu_1 > \mu_2$, there is no upper critical dimension in contrast to the case $\mu_1 = \mu_2$ where the upper critical dimension is $d_c = 2$. We have presented explicit forms of scaling functions for both the Takayasu line as well as the experimentally relevant uniform distribution case. These exact results will be useful for comparison with possible future experimental results on correlations in bead packs.

In this paper we have calculated exactly various time-dependent correlations between forces in bead packs in the context of the simple scalar q model. There have been various generalizations of this scalar model to include the tensorial nature of the forces [19,21] and also to noncohesive granular materials [22]. It would be interesting to see if our method can be extended to calculate the correlations in these generalized models.

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